

# Bosonization, Painlevé property, exact solutions for $\mathcal{N} = 1$ supersymmetric mKdV equation

Bo Ren<sup>1\*</sup>, Jian-Rong Yang<sup>2</sup>, Ping Liu<sup>3</sup>, Xi-Zhong Liu<sup>1</sup>

<sup>1</sup>*Institute of Nonlinear Science, Shaoxing University, Shaoxing 312000, China*

<sup>2</sup>*Department of Physics and Electronics, Shangrao Normal University, Shangrao 334001, China*

<sup>3</sup>*College of Electron and Information Engineering,*

*University of Electronic Science and Technology of China Zhongshan Institute, Zhongshan 528402, China*

(Dated: May 13, 2014)

The  $\mathcal{N} = 1$  supersymmetric modified Korteweg-de Vries (SmKdV) system is transformed to a system of coupled bosonic equations with the bosonization approach. The bosonized SmKdV (BSmKdV) passes the Painlevé test and allows a set of Bäcklund transformation (BT) by truncating the series expansions of the solutions about the singularity manifold. The traveling wave solutions of the BSmKdV system are obtained using the mapping and deformation method. Some special types of exact solutions for the BSmKdV system are found with the solutions and symmetries of the usual mKdV equation. In the meanwhile, the similarity reduction solutions of the system are investigated by using the Lie point symmetry theory. The generalized tanh function expansion method for the BSmKdV system leads to a nonauto-BT theorem. Using the nonauto-BT theorem, the novel exact explicit solutions of the BSmKdV system can be obtained. All these solutions obtained via the bosonization procedure are different from those obtained via other methods.

## I. INTRODUCTION

The mathematical formulation of supersymmetry is based on the introduction of Grassmann variables. It exists an extensive literature devoted to construction the symmsymmetric integrable models, such as Korteweg-de Vries [1], modified Korteweg-de Vries [2–5], Sine-Gordon [6], Kadomtsev-Petviashvili hierarchy [7] and nonlinear Schrödinger equation [8]. It has shown that these supersymmetric integrable systems possess the Painlevé property, the Lax representation, an infinite number of conservation laws, the Bäcklund and the Darboux transformations, bilinear forms and multi-soliton solutions [9–16]. However, to treat the integrable systems with fermions such as the supersymmetric integrable systems and pure integrable fermionic systems is much more complicated than to study the integrable pure bosonic systems [17]. It is significant if one can establish a proper bosonization procedure to deal with the supersymmetric systems. Recently, a simple bosonization approach to treat the super integrable systems has been proposed [18, 19]. The method can effectively avoid difficulties caused by intractable fermionic fields which are anti-commuting [18–21].

In this letter, we shall use the bosonization approach to the SmKdV system [2–5]. It reads

$$\Phi_t + D^6\Phi - 3\Phi D^3\Phi D\Phi - 3(D\Phi)^2 D^2\Phi = 0, \quad (1)$$

where  $D = \partial_\theta + \theta\partial_x$  is the covariant derivative. It is established with the usual independent variable  $\{x, t\}$  and a Grassmann variable  $\theta$ , and expansion  $\Phi$  in terms of  $\theta$  yields  $\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t)$ . (1) is related to the  $\mathcal{N} = 1$  supersymmetric KdV through a Miura type of transformation [4, 5] and has a bilinear Bäcklund transformation [10]. It shares the common conserved quantities with

---

\* Electronic mail: renbosemail@gmail.com.

the supersymmetric Sine-Gordon equation [13]. The quasi-periodic wave solutions are constructed with the Hirota bilinear method and the Riemann theta function recently [22].

The paper is organized as follows. In section 2, based on the bosonization approach, the  $\mathcal{N} = 1$  SmKdV system is changed to a system of coupled bosonic equations. The Painlevé property and the BT of the coupled bosonic equations are studied by the standard singularity analysis. In sections 3, some special types of exact solutions can be explicitly found by means of the mapping and deformation method. In sections 4, the reduction solutions for the usual Painlevé II are found using the Lie point symmetry. Section 5 is devoted to the generalized tanh function expansion approach for the coupled bosonic equations. The explicit novel exact solution of the BSmKdV is investigated. The last section is a simple summary and discussion.

## II. BOSONIZATION OF THE SMKDV EQUATION AND PAINLEVÉ ANALYSIS

### A. Bosonization approach with two fermionic parameters

In terms of the component fields, (1) is equivalent to

$$u_t + u_{xxx} - 6u^2u_x - 3\xi(u\xi_x)_x = 0, \quad (2a)$$

$$\xi_t + \xi_{xxx} - 3u u_x \xi - 3u^2 \xi_x = 0. \quad (2b)$$

It is obvious that (2) includes a commuting  $u$  and an anticommuting  $\xi$  field. It will degenerate to the usual classical system with vanishing the fermionic sector. In order to avoid the difficulties in dealing with the anticommutative fermionic field  $\xi$ , we expand the component fields  $\xi$  and  $u$  by introducing the two fermionic parameters [18–21]

$$u(x, t) = v + w\zeta_1\zeta_2, \quad (3a)$$

$$\xi(x, t) = p\zeta_1 + q\zeta_2, \quad (3b)$$

where  $\zeta_1$  and  $\zeta_2$  are two Grassmann parameters, while the coefficients  $v$ ,  $p$ ,  $q$  and  $w$  are four usual real or complex functions with respect to the spacetime variable  $\{x, t\}$ . Substituting (3) into the SmKdV system (2), we obtain

$$v_t + v_{xxx} - 6v^2v_x = 0, \quad (4a)$$

$$p_t + p_{xxx} - 3v^2p_x - 3p v v_x = 0, \quad (4b)$$

$$q_t + q_{xxx} - 3v^2q_x - 3q v v_x = 0, \quad (4c)$$

$$w_t + w_{xxx} - 6(v^2w)_x - 3v_x(pq)_x - 3v p q_{xx} - 3v q p_{xx} = 0. \quad (4d)$$

The above way is just the bosonic procedure for the SmKdV system (2) with two fermionic parameters (BSmKdV-2). (4a) is exactly the usual mKdV equation which has been widely studied [23–26]. (4b)-(4d) are linear homogeneous in  $p$ ,  $q$  and  $w$ , respectively. These pure bosonic systems can be easily solved theoretically. This is just one of the advantages of the bosonization approach.

### B. Painlevé analysis and Bäcklund transformations for the BSmKdV-2 system

In this part, we will study the Painlevé property and the BT of the BSmKdV-2 system. If all the movable singularities of its solutions are only poles, the model is called Painlevé integrable. In

order to perform the Painlevé analysis, the bosonic fields  $v, p, q, w$  expand about the singularity manifold  $\phi(x, t) = 0$  as

$$v = \sum_{j=0}^{\infty} v_j \phi^{j-\alpha_1}, \quad p = \sum_{j=0}^{\infty} p_j \phi^{j-\alpha_2}, \quad q = \sum_{j=0}^{\infty} q_j \phi^{j-\alpha_3}, \quad w = \sum_{j=0}^{\infty} w_j \phi^{j-\alpha_4}, \quad (5)$$

with  $\{v_j, p_j, q_j, w_j\}$  being arbitrary functions of  $\{x, t\}$ . From the leading order analysis result, the all constants  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are positive integers, i.e., 1, 1, 1, 2 respectively. Consequently, the recursion relations to determine the functions  $v_j, p_j, q_j$  and  $w_j$  can be obtained, the resonance values of  $j$  are given

$$j = -1, 0, 0, 0, 2, 2, 3, 4, 4, 4, 4, 5. \quad (6)$$

After the detailed calculations, the resonance conditions are satisfied identically because the functions  $v_j, p_j, q_j$  and  $w_j$  are all determined by twelve arbitrary functions  $\phi, p_0, q_0, w_0, p_2, q_2, v_3, v_4, p_4, q_4, w_4$  and  $w_5$ . From the above considerations we deduce that the BSmKdV-2 is really Painlevé integrable.

Using the standard truncated Painlevé expansion, the BT is

$$v = \frac{v_0}{\phi} + v_1, \quad p = \frac{p_0}{\phi} + p_1, \quad q = \frac{q_0}{\phi} + q_1, \quad w = \frac{w_0}{\phi^2} + \frac{w_1}{\phi} + w_2, \quad (7)$$

where  $v_0 = \pm \phi_x$ ,  $w_1 = (\phi_x w_{0,x} - \phi_{xx} w_0) \phi_x^{-2}$  and  $\{v_1, p_1, q_1, w_2\}$  satisfy BSmKdV-2 system. Besides, we find the fields  $\phi, p_0, q_0$  and  $w_0$  are the solutions of the following Schwarzian BSmKdV-2 system

$$\phi_t + \phi_{xxx} - \frac{3}{2} \frac{\phi_{xx}^2}{\phi_x} = 0, \quad (8a)$$

$$p_{0,t} + p_{0,xxx} + \frac{15}{4} p_{0,x} \frac{\phi_{xx}^2}{\phi_x^2} + \frac{3}{2} p_0 \frac{\phi_{xx} \phi_{xxx}}{\phi_x^2} - \frac{3}{2} p_{0,x} \frac{\phi_{xxx}}{\phi_x} - \frac{9}{4} p_0 \frac{\phi_{xx}^3}{\phi_x^3} - 3 p_{0,xx} \frac{\phi_{xx}}{\phi_x} = 0, \quad (8b)$$

$$q_{0,t} + q_{0,xxx} + \frac{15}{4} q_{0,x} \frac{\phi_{xx}^2}{\phi_x^2} + \frac{3}{2} q_0 \frac{\phi_{xx} \phi_{xxx}}{\phi_x^2} - \frac{3}{2} q_{0,x} \frac{\phi_{xxx}}{\phi_x} - \frac{9}{4} q_0 \frac{\phi_{xx}^3}{\phi_x^3} - 3 q_{0,xx} \frac{\phi_{xx}}{\phi_x} = 0, \quad (8c)$$

$$w_{0,t} + w_{0,xxx} + \frac{27}{2} w_{0,x} \frac{\phi_{xx}^2}{\phi_x^2} + 6 w_0 \frac{\phi_{xx} \phi_{xxx}}{\phi_x^2} - 12 w_0 \frac{\phi_{xx}^3}{\phi_x^3} - 3 w_{0,x} \frac{\phi_{xxx}}{\phi_x} - 6 w_{0,xx} \frac{\phi_{xx}}{\phi_x} + 3 \frac{q_0 p_{0,xx} \phi_{xx}}{\phi_x} - 3 \frac{p_0 q_{0,xx} \phi_{xx}}{\phi_x} + \frac{3}{2} \frac{p_0 q_0 \phi_{xxx}}{\phi_x} - \frac{3}{2} \frac{q_0 p_0 \phi_{xxx}}{\phi_x} + \frac{9}{4} \frac{p_0 q_0 \phi_{xx}^2}{\phi_x^2} - \frac{9}{4} \frac{q_0 p_0 \phi_{xx}^2}{\phi_x^2} + 3 p_{0,x} q_{0,xx} - 3 q_{0,x} p_{0,xx} = 0, \quad (8d)$$

with the solutions  $\phi, p_0, q_0$  and  $w_0$  are related by

$$\begin{aligned} v_1 &= -\frac{\phi_{xx}}{2\phi_x}, & p_1 &= \frac{p_0 \phi_{xx} - 2p_{0,x} \phi_x}{2\phi_x^2}, & q_1 &= \frac{q_0 \phi_{xx} - 2q_{0,x} \phi_x}{2\phi_x^2}, \\ w_2 &= \frac{1}{12\phi_x^4} (12q_0 \phi_{xx} p_{0,x} \phi_x - 12p_0 \phi_{xx} q_{0,x} \phi_x + 6p_0 q_{0,xx} \phi_x^2 - 6q_0 p_{0,xx} \phi_x^2 \\ &\quad - 2w_0 \phi_t \phi_x - 12p_0 \phi_x^4 q_2 + 12q_0 \phi_x^4 w_0 + 6w_{0,xx} \phi_x^2 - 18w_{0,x} \phi_x \phi_{xx} + 21w_0 \phi_{xx}^2). \end{aligned}$$

It is obvious that an auto-BT (7) and a nonauto-BT (8) are obtained with the singularity analysis.

### III. TRAVELING WAVE SOLUTIONS WITH MAPPING AND DEFORMATION METHOD

Now the traveling wave solutions of the bosonic (4) will be studied. Introducing the traveling wave variable  $X = kx + \omega t + c_0$  with constants  $k, \omega$  and  $c_0$ , (4) is transformed to the ordinary differential equations (ODEs)

$$k^3 v_{XXX} + \omega v_X - 6k v^2 v_X = 0, \quad (9a)$$

$$k^3 p_{XXX} + \omega p_X - 3kpv_X - 3kv^2 p_X = 0, \quad (9b)$$

$$k^3 q_{XXX} + \omega q_X - 3kqv_X - 3kv^2 q_X = 0, \quad (9c)$$

$$k^3 w_{XXX} + \omega w_X - 6k(v^2 w)_X + 3k^2 q(vp_X)_X - 3k^2 p(vq_X)_X = 0. \quad (9d)$$

As the well known exact solutions of (9a), we try to build the mapping and deformation relationship between the traveling wave solutions  $v$  and  $\{p, q, w\}$ , then the exact solutions of the BSmKdV-2 equation can be obtained with the known solutions of mKdV equation.

At first, we get  $v_X$  from (9a)

$$v_X = \frac{a_0 \sqrt{k(kv^4 - \omega v^2 - 2c_1 v + c_2 k^3)}}{k^2}, \quad (10)$$

where  $c_1$  and  $c_2$  are the integral constants and  $a_0^2 = 1$ . In order to get the mapping relationship between  $v$  and  $\{p, q, w\}$ , we introduce the variable transformations

$$p(X) = P(v(X)), \quad q(X) = Q(v(X)), \quad w(X) = W(v(X)). \quad (11)$$

Using the transformation (11) and vanishing  $v_X$  via (10), the linear ODEs (9b)-(9d) become

$$(kv^4 - \omega v^2 - 2c_1 v + c_2 k^3) \frac{d^3 P}{dv^3} + (6kv^3 - 3\omega v - 3c_1) \frac{d^2 P}{dv^2} + 3kv^2 \frac{dP}{dv} - 3kvP = 0, \quad (12a)$$

$$(kv^4 - \omega v^2 - 2c_1 v + c_2 k^3) \frac{d^3 Q}{dv^3} + (6kv^3 - 3\omega v - 3c_1) \frac{d^2 Q}{dv^2} + 3kv^2 \frac{dQ}{dv} - 3kvQ = 0, \quad (12b)$$

$$(kv^4 - \omega v^2 - 2c_1 v + c_2 k^3) \frac{d^2 W}{dv^2} + (2kv^3 - \omega v - c_1) \frac{dW}{dv} + (\omega - 6kv^2)W - F(v) = 0, \quad (12c)$$

where

$$F(v) = 3a_0 \sqrt{k(kv^4 - \omega v^2 - 2c_1 v + c_2 k^3)} \left( P \frac{dQ}{dv} - Q \frac{dP}{dv} \right) + c_3.$$

The mapping and deformation relations are constructed via (12)

$$P = A_1 v + (A_2 + A_3) v \cos(R(v)) - (A_2 - A_3) v \sin(R(v)), \quad (13a)$$

$$Q = A_4 v + (A_5 + A_6) v \cos(R(v)) - (A_5 - A_6) v \sin(R(v)), \quad (13b)$$

$$W = \left( A_7 + \int^v \frac{A_8 + yF(y)}{(ky^4 - \omega y^2 - 2c_1 y + c_2 k^3)^{3/2}} dy \right) \sqrt{kv^4 - \omega v^2 - 2c_1 v + c_2 k^3}, \quad (13c)$$

where  $A_k$  ( $k = 1, 2, \dots, 8$ ) are arbitrary constants, and

$$R(v) = \int^u \frac{ik\sqrt{c_2 k}}{y\sqrt{ky^4 - \omega y^2 - 2c_1 y + c_2 k^3}} dy.$$

If we know the solution of  $v$ , the traveling wave solution of BSmKdV-2 system will be given with considering (13) and (11). For a special case,  $A_8 = c_3 = 0$ ,  $A_2 = A_5$  and  $A_3 = A_6$ , the traveling wave solution  $W$  is an ordinary type of the symmetries of the traveling wave equation (9a). In fact, for any given a solution  $v$  of the usual mKdV equation, a special type solutions of the bosonic equation (4) can be constructed

$$p = P = A_1 v, \quad q = Q = A_4 v, \quad w = W = A_7 \sigma(v), \quad (14)$$

where  $\sigma(v)$  is the symmetry of the usual mKdV equation (9a). The field  $w$  exactly satisfy the symmetry equation of the usual mKdV system. The solution  $v$  is not restricted to the traveling wave solutions. We can construct not only traveling wave solutions but also some novel types of solutions of the BSmKdV-2 system by using the solutions and symmetries of the mKdV equation.

#### IV. SIMILARITY REDUCTION SOLUTIONS WITH LIE POINT SYMMETRY THEORY

It is well known that the Lie point symmetry play an important role in the investigation of nonlinear partial differential equations (PDEs) in modern mathematical physics. The approach is effective methods to obtain the explicit exact solutions [27–30]. Our aim is to apply the techniques of Lie group theory to the coupled bosonic equation in order to obtain particular exact solutions and to study their properties. First, we assume the corresponding Lie point symmetry has the vector form

$$V = X \frac{\partial}{\partial x} + T \frac{\partial}{\partial t} + V \frac{\partial}{\partial v} + P \frac{\partial}{\partial p} + Q \frac{\partial}{\partial q} + W \frac{\partial}{\partial w}, \quad (15)$$

where  $X$ ,  $T$ ,  $V$ ,  $P$ ,  $Q$  and  $W$  are functions with respect to  $x$ ,  $t$ ,  $v$ ,  $p$ ,  $q$  and  $w$ . The symmetry supposes

$$\sigma_0 = Xv_x + Tv_t - V, \quad \sigma_1 = Xp_x + Tp_t - P, \quad \sigma_2 = Xq_x + Tq_t - Q, \quad \sigma_3 = Xw_x + Tw_t - W. \quad (16)$$

The symmetry  $\sigma_k$  ( $k = 0, 1, 2, 3$ ) is the solution of the linearized equations of (4)

$$\sigma_{0,t} + \sigma_{0,xxx} - 6(v^2\sigma_0)_x = 0, \quad (17a)$$

$$\sigma_{1,t} + \sigma_{1,xxx} - 3vv_x\sigma_1 - 3v^2\sigma_{1,x} - 3p(\sigma_0v)_x - 6v\sigma_0p_x = 0, \quad (17b)$$

$$\sigma_{2,t} + \sigma_{2,xxx} - 3vv_x\sigma_2 - 3v^2\sigma_{2,x} - 3q(\sigma_0v)_x - 6v\sigma_0q_x = 0, \quad (17c)$$

$$\begin{aligned} \sigma_{3,t} + \sigma_{3,xxx} - 6(\sigma_3v^2)_x - 12(\sigma_0vw)_x + 3\sigma_2(vp_x)_x - 3\sigma_1(vq_x)_x \\ - 3p(\sigma_0q_x + \sigma_{2,x}v)_x + 3q(\sigma_0p_x + \sigma_{1,x}v)_x = 0. \end{aligned} \quad (17d)$$

Substituting (16) into the symmetry equations (17) and eliminating  $v$ ,  $p$ ,  $q$  and  $w$  in terms of (4), the solutions  $X$ ,  $T$ ,  $V$ ,  $P$ ,  $Q$  and  $W$  can be concluded using the determining equations

$$\begin{aligned} T = C_1t + C_2, \quad X = \frac{C_1x}{3} + C_7, \quad V = -\frac{C_1}{3}v, \\ P = C_3p + C_4q, \quad Q = C_6q + C_5p, \quad W = w(C_3 + C_6), \end{aligned} \quad (18)$$

where  $C_i$  ( $i = 1, 2, \dots, 7$ ) are arbitrary constants. Then, one can solve the characteristic equations

$$\frac{dx}{X} = \frac{dt}{T}, \quad \frac{dv}{V} = \frac{dt}{T}, \quad \frac{dp}{P} = \frac{dt}{T}, \quad \frac{dq}{Q} = \frac{dt}{T}, \quad \frac{dw}{W} = \frac{dt}{T}, \quad (19)$$

where  $X$ ,  $T$ ,  $V$ ,  $P$ ,  $Q$ , and  $W$  are given by (18). One case about the solution (4) will discuss in the following.

When  $C_1 = C_4 = C_5 = 0$ , we can find the similarity solutions after solving out the characteristic equations

$$v = V(\xi), \quad p = P(\xi)e^{\frac{C_3x}{C_7}}, \quad q = Q(\xi)e^{\frac{C_6x}{C_7}}, \quad w = W(\xi)e^{\frac{(C_3+C_6)x}{C_7}}, \quad (20)$$

with the similarity variable  $\xi = t - (C_2/C_7)x$ . Substituting (20) into (4), the invariant functions  $V$ ,  $P$ ,  $Q$  and  $W$  satisfy the reduction systems

$$V_{\xi\xi\xi} - \frac{C_7^3}{C_2^3}V_{\xi} - \frac{6C_7^2}{C_2^2}V^2V_{\xi} = 0, \quad (21a)$$

$$P_{\xi\xi\xi} - \frac{3C_3}{C_2}P_{\xi\xi} - \frac{3C_7^2}{C_2^2}P_{\xi}V^2 - \frac{3C_7^2}{C_2^2}P V V_{\xi} + \frac{3C_2C_3^2 - C_7^3}{C_2^3}P_{\xi}V^2 - \frac{C_3^3}{C_2^3}P + \frac{3C_3C_7^2}{C_2^3}P V^2 = 0, \quad (21b)$$

$$Q_{\xi\xi\xi} - \frac{3C_6}{C_2}Q_{\xi\xi} - \frac{3C_7^2}{C_2^2}Q_{\xi}V^2 - \frac{3C_7^2}{C_2^2}Q V V_{\xi} + \frac{3C_2C_6^2 - C_7^3}{C_2^3}Q_{\xi}V^2 - \frac{C_6^3}{C_2^3}Q + \frac{3C_6C_7^2}{C_2^3}Q V^2 = 0 \quad (21c)$$

$$\begin{aligned} W_{\xi\xi\xi} + \frac{C_3^3 - 3C_2^2C_6 - 3C_2^2C_3}{C_2^3}W_{\xi\xi} - \frac{6C_7^2}{C_2^2}(V^2W)_{\xi} - \frac{C_3^3 + C_6^3 + 3C_3C_6^2 + 3C_3^2C_6}{C_2^3}W \\ + \frac{6C_7^2(C_3 + C_6)}{C_2^3}V^2W + \frac{3C_2C_6^2 + 3C_2C_3^2 + 6C_2C_3C_6 - C_7^3}{C_2^3}W_{\xi} + \frac{3C_7(C_6^2 - C_3^2)}{C_2^3}V P Q \\ - \frac{3C_7}{C_2}Q(V P_{\xi})_{\xi} + \frac{3C_7}{C_2}P(V Q_{\xi})_{\xi} + \frac{3C_7(C_3 - C_6)}{C_2^2}V_{\xi}P Q + \frac{6C_7}{C_2^2}V(C_3P_{\xi}Q - C_6P Q_{\xi}) = 0. \end{aligned} \quad (21d)$$

which (21a) satisfies the Painlevé II equation. These reduction equations are linear ODEs while the previous functions are known, we can solve  $V$ ,  $P$ ,  $Q$ , and  $W$  one after another in principle. The group invariant solution of an interaction solution among a Painlevé II wave and a soliton is obtained with the Lie point symmetry theory.

## V. GENERALIZED TANH FUNCTION EXPANSION METHOD OF BSMKDV-2 SYSTEM

The truncated Painlevé expansion approach and the generalized tanh function expansion method are established to find interactions among different nonlinear excitations [31]. The methods are valid for all integrable systems and or even nonintegrable models because both the truncated Painlevé analysis and the tanh expansion method can be used to find exact solutions of the partially solvable nonlinear models [21, 31].

According to the usual tanh function expansion method, the generalized expansion solution has the form

$$v = v_0 + v_1 \tanh(f), \quad (22a)$$

$$p = p_0 + p_1 \tanh(f), \quad (22b)$$

$$q = q_0 + q_1 \tanh(f), \quad (22c)$$

$$w = w_0 + w_1 \tanh(f) + w_2 \tanh^2(f). \quad (22d)$$

where  $v_0$ ,  $v_1$ ,  $p_0$ ,  $p_1$ ,  $q_0$ ,  $q_1$ ,  $w_0$ ,  $w_1$ ,  $w_2$  and  $f$  are functions of  $\{x, t\}$  and should be determined later. After some detail calculations by substituting (22) into the BSmKdV-2 system (4), we can prove the following nonauto-BT theorem.

**Theorem (Nonauto-BT theorem).** If  $\{f, g, h, n\}$  is a solution of

$$f_t + f_{xxx} - 2f_x^3 - \frac{3}{2}\frac{f_{xx}^2}{f_x} = 0, \quad (23a)$$

$$g_t + g_{xxx} - 6g_x f_x^2 - g_{xx} \frac{3f_{xx}}{f_x} + g_x \frac{3f_t}{2f_x} + g_x \frac{3f_{xx}^2}{2f_x^2} - g \frac{3f_t f_{xx}}{2f_x^2} = 0, \quad (23b)$$

$$h_t + h_{xxx} - 6h_x f_x^2 - h_{xx} \frac{3f_{xx}}{f_x} + h_x \frac{3f_t}{2f_x} + h_x \frac{3f_{xx}^2}{2f_x^2} - h \frac{3f_t f_{xx}}{2f_x^2} = 0, \quad (23c)$$

$$\begin{aligned} n_t + n_{xxx} - n_{xx} \frac{6f_{xx}}{f_x} + n_x \left( \frac{9f_{xx}^2}{f_x^2} + \frac{3f_t}{f_x} - 12f_x^2 \right) - n \left( \frac{6f_t f_{xx}}{f_x^2} + \frac{3f_{xx}^3}{f_x^3} - 12f_x f_{xx} \right) + 6f_x^2 (gh_x - hg_x) \\ + \frac{3f_t}{2f_x} (hg_x - gh_x) + \frac{9f_{xx}^2}{2f_x^2} (hg_x - gh_x) + \frac{3f_{xx}^2}{f_x} (hg_{xx} - 3gh_{xx}) + 3g_x h_{xx} - 3h_x g_{xx} = 0. \end{aligned} \quad (23d)$$

then  $\{v, p, q, w\}$  with

$$v = -\frac{f_{xx}}{2f_x} + f_x \tanh(f), \quad (24a)$$

$$p = \frac{f_{xx}}{2f_x^2}g - \frac{g_x}{f_x} + g \tanh(f), \quad (24b)$$

$$q = \frac{f_{xx}}{2f_x^2}h - \frac{h_x}{f_x} + h \tanh(f), \quad (24c)$$

$$w = \frac{n_{xx}}{2f_x^2} - n_x \frac{3f_{xx}}{2f_x^3} + n \left( \frac{f_t}{2f_x^3} + \frac{3f_{xx}^2}{4f_x^4} - 2 \right) + \frac{1}{2f_x^2} (gh_{xx} - hg_{xx}) + \frac{f_{xx}}{f_x^3} (hg_x - gh_x) \\ + \left( \frac{f_{xx}n}{f_x^2} - \frac{n_x}{f_x} \right) \tanh(f) + n \tanh^2(f). \quad (24d)$$

is a solution of the BSmKdV-2 system (4).

We can find some nontrivial solutions of the BSmKdV-2 from some quite trivial solutions of (23). Here we list an interesting examples. A quite trivial straight-line solution of (23) has the form

$$f = k_0x + \omega_0t + l_0, \quad g = k_1x + \omega_1t + l_1, \quad h = k_2x + \omega_2t + l_2, \quad n = k_3x + \omega_3t + l_3, \\ \omega_0 = 2k_0^3, \quad \omega_1 = 3k_0^2k_1, \quad \omega_2 = 3k_0^2k_2, \quad \omega_3 = 3k_0^2(2k_3 + k_1l_2 - k_2l_1), \quad (25)$$

where  $k_0, k_1, k_2, k_3, l_0, l_1, l_2$  and  $l_3$  are all the free constants. Substituting the line solution (25) into the nonauto-BT theorem yields the following soliton solution of BSmKdV-2 system

$$v = k_0 \tanh(f), \quad (26a)$$

$$p = g \tanh(f) - \frac{k_1}{k_0}, \quad (26b)$$

$$q = h \tanh(f) - \frac{k_2}{k_0}, \quad (26c)$$

$$w = n \tanh^2(f) - \frac{k_3}{k_0} \tanh(f) - n. \quad (26d)$$

Though the soliton solution (26) is a traveling wave in the space time  $\{x, t\}$  for the boson field  $v$ , it is not a traveling wave for other boson fields  $p, q$  and  $w$ , then the superfield  $\Phi$  of SmKdV is not a traveling wave except for the case of  $g, h$  and  $n$  being constants, i.e.,  $k_1 = k_2 = k_3 = 0$ . This example reveals that an straightening the single soliton to a straight-line solution for the BSmKdV-2 is given by the nonauto-BT theorem.

## VI. CONCLUSIONS

In summary, the bosonization procedure has been successfully applied to the SmKdV equation. The SmKdV equation is simplified to the mKdV equation together with three linear differential equations. The BSmKdV-2 system is proved to possess Painlevé property and to be completely integrable. The auto-BT and nonauto-BT are constructed by truncating the standard Painlevé expansion.

The traveling wave solutions are studied by using the mapping and deformation method. Some special types of exact solutions can be given straightforwardly through the exact solutions of the mKdV equation and its symmetries. In addition, the group invariant solutions of the system are

derived with the Lie point symmetry method. The generalized tanh function expansion method is developed to find interaction solutions among different nonlinear excitations. Straightening a single soliton to a straight-line solution for the BSmKdV-2 system is constructed with the generalized tanh function expansion method. Using the nonauto-BT theorem, various exact explicit solutions of the BSmKdV-2 system can be obtained. All these solutions obtained via the bosonization procedure are different from those obtained via other methods such as the Hirota bilinear method and the Riemann theta function [10, 22].

In this paper, the properties and exact solutions of the BSmKdV-2 system are investigated, we can also introduce  $N$  fermionic parameters to expand the SmKdV system (BSmKdV-N). For the  $N \geq 2$  fermionic parameters  $\zeta_i$  ( $i = 1, 2, \dots, N$ ) instance, the component fields  $u$  and  $\xi$  expand

$$\xi(x, t) = \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} \sum_{1 \leq i_1 < \dots < i_{2n-1} \leq N} u_{i_1 i_2 \dots i_{2n-1}} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_{2n-1}}, \quad (27a)$$

$$u(x, t) = u_0 + \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} \sum_{1 \leq i_1 < \dots < i_{2n} \leq N} u_{i_1 i_2 \dots i_{2n}} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_{2n}}, \quad (27b)$$

where the coefficients  $u_0, u_{i_1 i_2 \dots i_{2n}}$  ( $1 \leq i_1 < \dots < i_{2n} \leq N$ ) and  $u_{i_1 i_2 \dots i_{2n-1}}$  ( $1 \leq i_1 < \dots < i_{2n-1} \leq N$ ) are  $2^N$  real or complex bosonic functions of classical spacetime variable  $\{x, t\}$ . The Painlevé property and exact solutions are worthy study for the BSmKdV-N system.

#### Acknowledgment:

We would like to thank S. Y. Lou for useful discussions. This work was partially supported by the National Natural Science Foundation of China under Grant (Nos. 11305106, 11365017 and 11305031), the Natural Science Foundation of Zhejiang Province of China under Grant (No. LQ13A050001) and the Natural Science Foundation of Guangdong Province (No. S2013010011546).

- 
- [1] B. A. Kupershmidt, Phys. Lett. A **102**, 213 (1984)
  - [2] M. Gurses and O. Oguz, Phys. Lett. A **108**, 437 (1985)
  - [3] M. Gurses and O. Oguz, Lett. Math. Phys. **11**, 235 (1986)
  - [4] P. Mathieu, J. Math. Phys. **29**, 2499 (1988)
  - [5] I. Yamanaka and R. Sasaki, Prog. Theor. Phys. **79**, 1167 (1988)
  - [6] J. Hruby, Nucl. Phys. B **131**, 275 (1977); S. Ferrara, L. Girardello and S. Sciuto, Phys. Lett. B **76**, 303 (1978)
  - [7] Y. I. Martin and A. O. Radul, Commun. Math. Phys. **98**, 65 (1985)
  - [8] G. H. M. Roelofs, and P. H. M. Kersten, J. Math. Phys. **33**, 2185 (1992); J. C. Brunelli and A. Das, *ibid.* **36**, 268 (1995)
  - [9] M. Chaichan and P. P. Kulish, Phys. Lett. B **78**, 413 (1978)
  - [10] Q. P. Liu, X. B. Hu and M. X. Zhang, Nonlinearity **18**, 1597 (2005)
  - [11] Q. P. Liu, Lett. Math. Phys. **35**, 115 (1995)
  - [12] Q. P. Liu and Y. F. Xie, Phys. Lett. A **325**, 139 (2004)
  - [13] I. Yamanaka and R. Sasaki, **79**, 1167 (1988)
  - [14] P. H. M. Kersten, Phys. Lett. A **134**, 25 (1988)
  - [15] S. Bellucci, E. Ivanov, S. Krivonos and A. Pichugin, Phys. Lett. B **312**, 463 (1993)
  - [16] A. S. Carstea, Nonlinearity **13**, 1645 (2000)
  - [17] M. S. Plyushchay, Ann. Phys. **245**, 339 (1996); F. Correa and M. S. Plyushchay, Ann. Phys. **322**, 2493 (2007)



- [18] S. Andrea, A. Restuccia and A. Sotomayor, J. Math. Phys. **42**, 2625 (2001)
- [19] X. N. Gao and S. Y. Lou, Phys. Lett. B **707**, 209 (2012)
- [20] B. Ren, J. Lin and J. Yu, AIP Advances **3**, 042129 (2013)
- [21] X. N. Gao, S. Y. Lou and X. Y. Tang, JHEP **05**, 029 (2013)
- [22] L. Luo and E. Fan, Nonlinear Anal. **74**, 666 (2011)
- [23] S. Tanaka, Proc. Japan Acad. **48**, 466 (1972)
- [24] R. Hirota, J. Phys. Soc. Jpn. **33**, 1456 (1972)
- [25] M. Wadati, J. Phys. Soc. Jpn. **32**, 1681 (1972)
- [26] Z. Y. Yan, Commun. Nonlinear Sci. Numer. Simul. **4**, 284 (1999)
- [27] P. J. Olver, *Application of Lie group to differential equation* (Springer, Berlin, 1986); G. W. Bluman and S. C. Anco, *Symmetry and integration methods for differential equations* (Springer, New York, 2002).
- [28] B. Ren, X. J. Xu and J. Lin, J. Math. Phys. **50**, 123505 (2009)
- [29] B. Li, C. Wang and Y. Chen, J. Math. Phys. **49**, 103503 (2008)
- [30] B. Ren, J. Y. Wang, J. Yu and J. Lin, Chin. J. Phys. **51**, 657 (2013)
- [31] S. Y. Lou, X. P. Cheng and X. Y. Tang, arXiv:1208.5314 [nlin.SI]